Adinkras as Origami

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Abstract

Around 20 years ago, physicists Michael Faux and Jim Gates invented Adinkras as a way to better understand Supersymmetry. These are bipartite graphs whose vertices represent bosons and fermions, and whose edges represent operators which relate the particles. Recently, Doran et al. determined that Adinkras are a type of Dessin d'Enfant by explicitly exhibiting a Belyĭ map as a composition $\beta : S \to \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. We are interested in exhibiting the same Belyĭ map as a different composition $\beta : S \to E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$.

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Adinkras

Let $\mathbb{F}_2 = \{0, 1\}$ be the finite field of 2 elements. Fix an integer $n \geq 2$. Denote $\mathbb{F}_2^n$ as the n-dimensional vector space over this field, where a vector $v = (v_1, v_2, \ldots, v_n)$ has components $v_i \in \mathbb{F}_2$.

An Adinkra is a bipartite graph constructed as follows. Define $\mathrm{ht}\colon \mathbb{F}_2^n \to \mathbb{Z}$ via counting the number of non zero components v_i of v. Choose a subspace $C \subseteq \mathrm{ht}^{-1}(4\mathbb{Z})$; elements are called doubly even codes. Construct a graph with "black" vertices $B = \mathrm{ht}^{-1}(2\mathbb{Z})/C$, "white" vertices $W = \mathrm{ht}^{-1}(2\mathbb{Z}+1)/C$, and edges $E = \{(v, w) \in \mathbb{F}_2^n \times \mathbb{F}_2^n \colon \mathrm{ht}(v - w) = 1\}/C$. It has the following properties:

1. It is an n-regular, bipartite graph whose faces are rectangular.
2. There are $|B| + |W| = 2^{n-m}$ vertices, $|F| = 2^{n-m-2} \cdot n$ faces, and $|E| = 2^{n-m-1} \cdot n$ edges, where $|C| = 2^m$.
3. $|E| = |B| + |W| + |F| + (2g - 2)$ where $g = 1 + 2^{n-m-3} \cdot (n - 4)$.

Examples of Adinkras

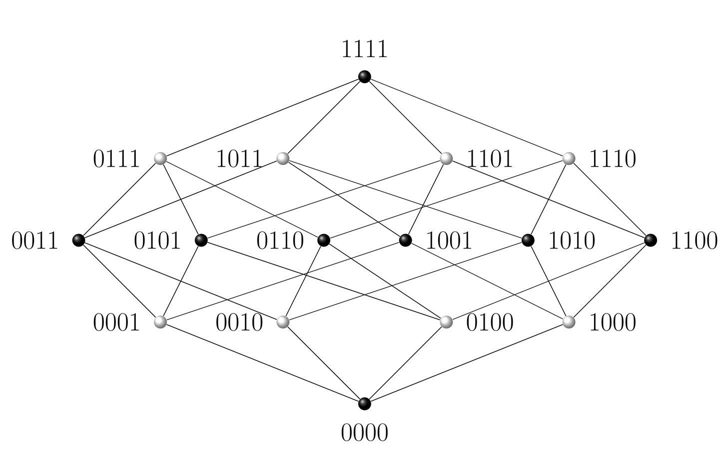


Figure 1. Adinkra corresponding to $n = 4$, $C = \{0000\}$

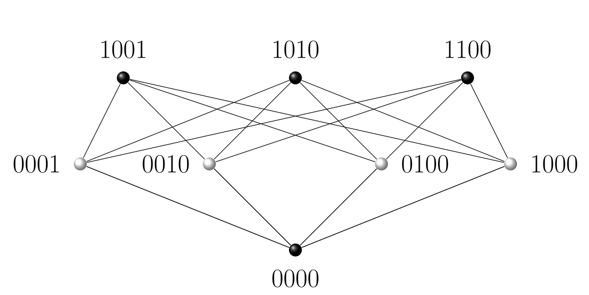


Figure 2. Adinkra corresponding to $n = 4$, $C = \{0000, 1111\}$

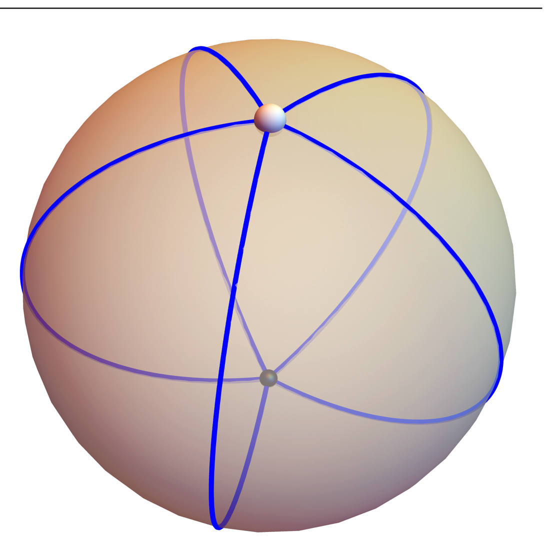
Example of a Belyĭ Map

For any positive integer n, consider the map $\widetilde{\beta}\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ given by

$$\widetilde{\beta}(z) = \frac{z^n}{z^n + 1}.$$

This is a $\widetilde{\beta}$ is a Belyĭ map of degree n.

The corresponding Dessin d'Enfant has one "black" vertex $B = \{0\}$, one "white" vertex $W = \{\infty\}$, $|E| = n$ edges, and $|F| = n$ faces.



Ramification Indices

Given a nonconstant map $\phi\colon S \to T$ between compact, connected Riemann surfaces S and T, the ramification index $e_\phi(P)$ at a point $P \in S$ is a natural number that effectively measures how much ϕ fails to be a covering map at P. We can describe the index by the following key properties.

1. The value $e_\phi(P) = 1$ for all but only finitely many $P \in S$.
2. For every point $Q \in T$, the degree of the map $\phi\colon S \to T$ is
$$\deg(\phi) = \sum_{P \in \phi^{-1}(Q)} e_\phi(P).$$
3. Say $\beta = \eta \circ \phi$ for some nonconstant maps $\phi\colon S \to T$ and $\eta\colon T \to T'$. Then we have the product $e_\beta(P) = e_\phi(P)\, e_\eta(\phi(P))$ for all points $P \in S$. Additionally, we have the product $\deg \beta = (\deg \phi)\,(\deg \eta)$.
4. Denote the genera of S and T as $g(S)$ and $g(T)$, respectively. Then
$$2\, g(S) - 2 = (\deg \phi)\, \big(2\, g(T) - 2\big) + \sum_{P \in S} \big(e_\phi(P) - 1\big).$$
5. Assume $\beta\colon S \to \mathbb{P}^1(\mathbb{C})$ is a Belyĭ map. The ramification indices $e_\beta(P) = 1$ whenever $q = \beta(P) \neq 0, 1, \infty$. Whenever $P \in \beta^{-1}(\{0, 1\})$, the indices $e_\beta(P)$ correspond to the number of edges incident to each vertex on the Dessin d'Enfant.

Examples of Adinkras as Belyĭ maps

Consider $n = 4$ and the subspace $C = \{0000\}$, which has dimension $m = 0$. We form an Adinkra from the elliptic curve $E : y^2 = x^3 - x$.

$$S = E(\mathbb{C}) \qquad P = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Big\downarrow [1+\sqrt{-1}] \qquad \Big\downarrow$$

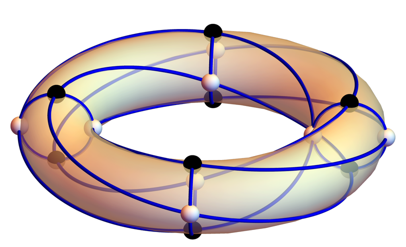
$$S = E(\mathbb{C}) \qquad Q = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} i\, \dfrac{x^2 - 1}{2\,x} \\ (-1+i)\, \dfrac{x^2+1}{4\,x^2}\, y \end{bmatrix}$$

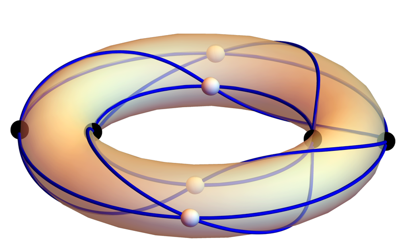
$$\Big\downarrow \qquad \Big\downarrow$$

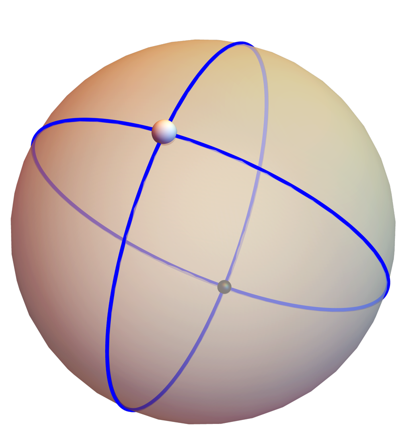
$$\mathbb{P}^1(\mathbb{C}) \qquad z = \frac{1+i}{\sqrt{2}}\, \frac{u^2+1}{2\,v} = \frac{i}{2\sqrt{2}}\, \frac{x^4 - 6\,x^2 + 1}{(x^2+1)\,y}$$

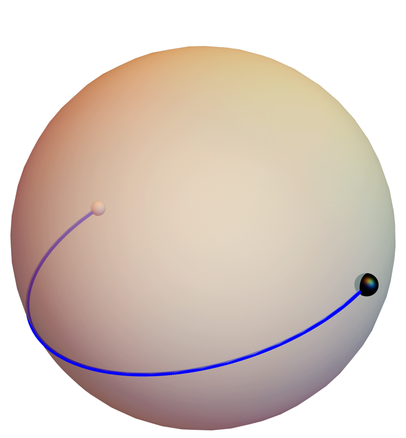
$$\Big\downarrow \widetilde{\beta} \qquad \Big\downarrow$$

$$\mathbb{P}^1(\mathbb{C}) \qquad q = \frac{z^4}{z^4+1} = \frac{(u^2+1)^4}{(u^4 - 6\,u^2 + 1)^2} = \frac{(x^4 - 6\,x^2 + 1)^4}{(x^8 + 20\,x^6 - 26\,x^4 + 20\,x^2 + 1)^2}$$









Belyĭ Maps and Dessins d'Enfants

Every compact, connected Riemann surface S is a smooth curve, that is, can be defined by a single polynomial

$$f(x, y) = \sum_{i,j} a_{ij}\, x^i y^j.$$

A Belyĭ map is a rational function $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which has critical values $q \in \{0, 1, \infty\}$, that is, $q = \beta(P)$ for some point $P = (x_0, y_0)$ which satisfies

$$f(P) = 0 \qquad \text{and} \qquad \frac{\partial \beta}{\partial x}(P)\, \frac{\partial f}{\partial y}(P) - \frac{\partial \beta}{\partial y}(P)\, \frac{\partial f}{\partial x}(P) = 0.$$

A Dessin d'Enfant is a bipartite graph on S corresponding to the preimage of $[0, 1] \subseteq \mathbb{P}^1(\mathbb{C})$ under a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$. Some properties are:

1. The "black" vertices correspond to $B = \beta^{-1}(0)$, "white" vertices to $W = \beta^{-1}(1)$, and faces to $F = \beta^{-1}(\infty)$.
2. The edges correspond to $E = \beta^{-1}([0, 1])$. In fact, the number of edges is the degree of the Belyĭ map, namely $|E| = |B| + |W| + |F| + (2\,g - 2)$, where g is the genus of the Riemann surface S.

Adinkras as Dessins d'Enfant

Doran et al. [2] proved the following: For an integer $n \geq 2$, fix a primitive $2n$th root of unity ζ. Let $\sigma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be that Möbius transformation such that $\sigma(\zeta) = 0$, $\sigma(\zeta^3) = 1$, and $\sigma(\zeta^{2n-1}) = \infty$.

1. The compact connected Riemann surface
$$S = \left\{ (x_1 : x_2 : \cdots : x_n) \in \mathbb{P}^{n-1}(\mathbb{C}) \;\middle|\; \begin{matrix} \sigma(\zeta^{2k-1})\, x_1^2 + x_2^2 + x_{k+1}^2 = 0 \\ \text{for } k = 2, 3, \ldots, n-1 \end{matrix} \right\}$$
has genus $g(S) = 1 + 2^{n-3} \cdot (n - 4)$.
2. There exists a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which sends
$$P = (x_1 : \cdots : x_n) \quad \mapsto \quad z = \sigma^{-1}\left(-\frac{x_2^2}{x_1^2}\right) \quad \mapsto \quad \frac{z^n}{z^n + 1}.$$
Its Dessin d'Enfant has $|B| = 2^{n-1}$ "black" vertices, $|W| = 2^{n-1}$ "white" vertices, $|E| = 2^{n-1} \cdot n$ edges, and $|F| = 2^{n-2} \cdot n$ rectangular faces.
3. Every Adinkra can be constructed using the Belyĭ pair (S, β).

PRiME 2023 Motivating Question

Doran et al. construct $\beta = \widetilde{\beta} \circ \varphi$, where $\widetilde{\beta} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ describes the "coloring of the edges" of the Adinkra. Can we also write $\beta = \eta \circ \phi$ where $\phi : S \to E(\mathbb{C})$ describes the "rectangular" nature of the faces?

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & \mathbb{P}^1(\mathbb{C}) \\ \downarrow \phi & & \downarrow \widetilde{\beta} \\ E(\mathbb{C}) & \xrightarrow{\eta} & \mathbb{P}^1(\mathbb{C}) \end{array} \qquad \begin{array}{ccc} P = (x_1 : x_2 \cdots : x_n) & \longrightarrow & z = \sigma^{-1}\left(-\dfrac{x_2^2}{x_1^2}\right) \\ \downarrow & & \downarrow \\ Q = (x, y) & \longrightarrow & q = \eta(Q) = \dfrac{z^n}{z^n + 1} \end{array}$$

What can we say about $\eta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$? How do we find E?

PRiME 2023 Theorem 1

Consider the Belyĭ pair (S, β) as in Doran et al.

1. For integers r and s satisfying $1 < r < s < n$, the quadric intersection
$$E(\mathbb{C}) = \left\{ (x_1 : x_2 : x_{r+1} : x_{s+1}) \in \mathbb{P}^3(\mathbb{C}) \;\middle|\; \begin{matrix} \sigma(\zeta^{2r-1})\, x_1^2 + x_2^2 + x_{r+1}^2 = 0 \\ \sigma(\zeta^{2s-1})\, x_1^2 + x_2^2 + x_{s+1}^2 = 0 \end{matrix} \right\}$$
is an elliptic curve which has j-invariant
$$j(E) = 256\, \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2\, (\lambda - 1)^2} \quad \text{in terms of} \quad \lambda = \frac{\sigma(\zeta^{2r-1})}{\sigma(\zeta^{2r-1}) - \sigma(\zeta^{2s-1})}.$$
2. The Belyĭ map $\beta = \eta \circ \phi$ in terms of that Toroidal Belyĭ map η which sends $Q = (x, y)$ to $q = z^n/(z^n + 1)$ in terms of
$$z = \frac{(x^2 - 2\,x + \lambda)^2 - \zeta\, \tau\, (x^2 - \lambda)^2}{\zeta\, (x^2 - 2\,x + \lambda)^2 - \tau\, (x^2 - \lambda)^2} \quad \text{where} \quad \tau = \sin \frac{q\pi}{n} \Big/ \sin \frac{(q-1)\pi}{n}.$$

Origami

Let $E : y^2 = x^3 + A\,x + B$ be an elliptic curve; recall that $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$ is a rectangle. A nonconstant morphism $\phi : S \to E(\mathbb{C})$ whose branch points $Q \in \{O_E\}$ is said to be an origami. Its degree is the integer

$$N = \sum_{P \in V} e_\phi(P) = |V| + \big(2\,g(S) - 2\big) \qquad \text{where} \qquad V = \phi^{-1}(O_E).$$

We may tile S by N squares having a total of $2\,N$ edges, where $P \in V$ are the vertices. For example, is $S = E'(\mathbb{C})$ is another elliptic curve, then $e_\phi(P) = 1$ so that $\phi : E' \to E$ is unbranched; this is an N-isogeny.

PRiME 2023 Theorem 2

Consider the Belyĭ pair (S, β) as in Doran et al. Assume that $\beta = \eta \circ \phi$ for some nonconstant maps $\eta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and $\phi : S \to E(\mathbb{C})$.

1. η must be a Toroidal Belyĭ map.
2. ϕ cannot be an origami whenever $n \geq 6$.

Future Work

- Adinkras are constructed from subspaces $C \subseteq \mathbb{F}_2^n$; they are quotients of the hypercube. We know that they can be embedded on a compact, connected Riemann surface of genus $g(S) = 1 + 2^{n-m-3} \cdot (n - 4)$. Find explicit embeddings when $n \geq 5$.
- The Belyĭ map $\eta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ in Theorem 1 has degree $\deg \eta = 8\,n$. Factor $\eta = \lambda \circ \gamma$ for (a) some $\gamma : E(\mathbb{C}) \to E'(\mathbb{C})$ with $\deg \gamma = 8$ and (b) some Toroidal Belyĭ map $\lambda : E'(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of $\deg \lambda = n$ whose Dessin d'Enfant has exactly one "black" vertex and one "white" vertex.

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